

APPENDIX TO V. MATHAI AND J. ROSENBERG'S PAPER "A NONCOMMUTATIVE SIGMA-MODEL"

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This short note is an appendix to [6].

Let $\theta \in \mathbb{R}$. Denote by A_θ the rotation C^* -algebra generated by unitaries U and V subject to $UV = e^{2\pi i\theta}VU$, and by A_θ^∞ its canonical smooth subalgebra. Denote by tr the canonical faithful tracial state on A_θ determined by $\text{tr}(U^m V^n) = \delta_{m,0}\delta_{n,0}$ for all $m, n \in \mathbb{Z}$. Denote by δ_1 and δ_2 the unbounded closed $*$ -derivations of A_θ defined on some dense subalgebras of A_θ and determined by $\delta_1(U) = 2\pi iU$, $\delta_1(V) = 0$, and $\delta_2(U) = 0$, $\delta_2(V) = 2\pi iV$. The *energy* [9], $E(u)$, of a unitary u in A_θ is defined as

$$(1) \quad E(u) = \frac{1}{2} \text{tr}(\delta_1(u)^* \delta_1(u) + \delta_2(u)^* \delta_2(u))$$

when u belongs to the domains of δ_1 and δ_2 , and ∞ otherwise.

Rosenberg has the following conjecture [9, Conjecture 5.4].

Conjecture 1. For any $m, n \in \mathbb{Z}$, in the connected component of $U^m V^n$ in the unitary group of A_θ^∞ , the functional E gets minimal value exactly at the scalar multiples of $U^m V^n$.

For a $*$ -endomorphism φ of A_θ^∞ , its *energy* [6], $\mathcal{L}(\varphi)$, is defined as $2E(\varphi(U)) + 2E(\varphi(V))$. Mathai and Rosenberg's Conjecture 3.1 in [6] about the minimal value of $\mathcal{L}(\varphi)$ follows directly from Conjecture 1.

Denote by H the Hilbert space associated to the GNS representation of A_θ for tr , and denote by $\|\cdot\|_2$ its norm. We shall identify A_θ as a subspace of H as usual. Then (1) can be rewritten as

$$E(u) = \frac{1}{2} (\|\delta_1(u)\|_2^2 + \|\delta_2(u)\|_2^2).$$

Now we prove Conjecture 1, and hence also prove Conjecture 3.1 of [6].

Theorem 2. Let $\theta \in \mathbb{R}$ and $m, n \in \mathbb{Z}$. Let $u \in A_\theta$ be a unitary whose class in $K_1(A_\theta)$ is the same as that of $U^m V^n$. Then $E(u) \geq E(U^m V^n)$, and “=” holds if and only if u is a scalar multiple of $U^m V^n$.

Proof. We may assume that u belongs to the domains of δ_1 and δ_2 . Set $a_j = u^* \delta_j(u)$ for $j = 1, 2$. For any closed $*$ -derivation δ defined on a dense subset of a unital C^* -algebra A and any tracial state τ of A vanishing on the range of δ , if unitaries v_1

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and v_2 in the domain of δ have the same class in $K_1(A)$, then $\tau(v_1^*\delta(v_1)) = \tau(v_2^*\delta(v_2))$ [7, page 281]. Thus

$$\mathrm{tr}(a_j) = \mathrm{tr}((U^m V^n)^* \delta_j(U^m V^n)) = \begin{cases} 2\pi i m & \text{if } j = 1; \\ 2\pi i n & \text{if } j = 2. \end{cases}$$

We have

$$\begin{aligned} \|\delta_j(u)\|_2^2 &= \|a_j\|_2^2 = \|\mathrm{tr}(a_j)\|_2^2 + \|a_j - \mathrm{tr}(a_j)\|_2^2 \\ &\geq \|\mathrm{tr}(a_j)\|_2^2 = |\mathrm{tr}(a_j)|^2 \\ &= \begin{cases} 4\pi^2 m^2 & \text{if } j = 1; \\ 4\pi^2 n^2 & \text{if } j = 2, \end{cases} \end{aligned}$$

and “=” holds if and only if $a_j = \mathrm{tr}(a_j)$. It follows that $E(u) \geq 2\pi^2(m^2 + n^2)$, and “=” holds if and only if $\delta_1(u) = 2\pi i m u$ and $\delta_2(u) = 2\pi i n u$. Now the theorem follows from the fact that the elements a in A_θ satisfying $\delta_1(a) = 2\pi i m a$ and $\delta_2(a) = 2\pi i n a$ are exactly the scalar multiples of $U^m V^n$. \square

When $\theta \in \mathbb{R}$ is irrational, the C^* -algebra A_θ is simple [10, Theorem 3.7], has real rank zero [1, Theorem 1.5], and is an AT-algebra [5, Theorem 4]. It is a result of Elliott that for any pair of AT-algebras with real rank zero, every homomorphism between their graded K -groups preserving the graded dimension range is induced by a $*$ -homomorphism between them [4, Theorem 7.3]. The graded dimension range of a unital simple AT-algebra A is the subset $\{(g_0, g_1) \in K_0(A) \oplus K_1(A) : 0 \not\leq g_0 \leq [1_A]_0\} \cup (0, 0)$ of the graded K -group $K_0(A) \oplus K_1(A)$ [8, page 51]. It follows that, when θ is irrational, for any group endomorphism ψ of $K_1(A_\theta)$, there is a unital $*$ -endomorphism φ of A_θ inducing ψ on $K_1(A_\theta)$. It is an open question when one can choose φ to be smooth in the sense of preserving A_θ^∞ , though it was shown in [2, 3] that if θ is irrational and φ restricts to a $*$ -automorphism of A_θ^∞ , then ψ must be an automorphism of the rank-two free abelian group $K_1(A_\theta)$ with determinant 1. When ψ is the zero endomorphism, from Theorem 2 one might guess that $\mathcal{L}(\varphi)$ could be arbitrarily small. It is somehow surprising, as we show now, that in fact there is a common positive lower bound for $\mathcal{L}(\varphi)$ for all $0 < \theta < 1$. This answers a question Rosenberg raised at the Noncommutative Geometry workshop at Oberwolfach in September 2009.

Theorem 3. *Suppose that $0 < \theta < 1$. For any unital $*$ -endomorphism φ of A_θ , one has $\mathcal{L}(\varphi) \geq 4(3 - \sqrt{5})\pi^2$.*

Theorem 3 is a direct consequence of the following lemma.

Lemma 4. *Let $\theta \in \mathbb{R}$ and let u, v be unitaries in A_θ with $uv = \lambda vu$ for some $\lambda \in \mathbb{C} \setminus \{1\}$. Then $E(u) + E(v) \geq 2(3 - \sqrt{5})\pi^2$.*

Proof. We have

$$\mathrm{tr}(uv) = \mathrm{tr}(\lambda vu) = \lambda \mathrm{tr}(uv),$$

and hence $\text{tr}(uv) = 0$. Thus

$$\begin{aligned} -\text{tr}(u)\text{tr}(v) &= \text{tr}(uv - \text{tr}(u)\text{tr}(v)) = \text{tr}((u - \text{tr}(u))v) + \text{tr}(\text{tr}(u)(v - \text{tr}(v))) \\ &= \text{tr}((u - \text{tr}(u))v). \end{aligned}$$

We may assume that both u and v belong to the domains of δ_1 and δ_2 . For any $m, n \in \mathbb{Z}$, denote by $a_{m,n}$ the Fourier coefficient $\langle u, U^m V^n \rangle$ of u . Then $a_{0,0} = \text{tr}(u)$, and

$$\begin{aligned} (2\pi)^2 \|u - \text{tr}(u)\|_2^2 &= \sum_{m,n \in \mathbb{Z}, m^2+n^2 > 0} |2\pi a_{m,n}|^2 \\ &\leq \sum_{m,n \in \mathbb{Z}, m^2+n^2 > 0} |2\pi a_{m,n}|^2 (m^2 + n^2) \\ &= \|\delta_1(u)\|_2^2 + \|\delta_2(u)\|_2^2 = 2E(u). \end{aligned}$$

Thus

$$|\text{tr}(u)|^2 = \|\text{tr}(u)\|_2^2 = \|u\|_2^2 - \|u - \text{tr}(u)\|_2^2 \geq 1 - \frac{1}{2\pi^2} E(u),$$

and

$$|\text{tr}((u - \text{tr}(u))v)| \leq \|(u - \text{tr}(u))v\|_2 = \|u - \text{tr}(u)\|_2 \leq \left(\frac{1}{2\pi^2} E(u)\right)^{1/2}.$$

Similarly, $|\text{tr}(v)|^2 \geq 1 - \frac{1}{2\pi^2} E(v)$.

Write $\frac{1}{2\pi^2} E(u)$ and $\frac{1}{2\pi^2} E(v)$ as t and s respectively. We just need to show that $t + s \geq 3 - \sqrt{5}$. If $t \geq 1$ or $s \geq 1$, then this is trivial. Thus we may assume that $1 - t, 1 - s > 0$. Then

$$(1 - t)(1 - s) \leq |\text{tr}(u)\text{tr}(v)|^2 \leq t.$$

Equivalently, $t(1 - s) \geq 1 - (t + s)$. Without loss of generality, we may assume $s \geq t$. Write $t + s$ as w . Then

$$t(1 - w/2) \geq t(1 - s) \geq 1 - (t + s) = 1 - w,$$

and hence

$$w = t + s \geq \frac{1 - w}{1 - w/2} + \frac{w}{2}.$$

It follows that $w^2 - 6w + 4 \leq 0$. Thus $w \geq 3 - \sqrt{5}$. □

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